A First Course in

MATHEMATICAL MODELING

Fifth Edition

Frank R. Giordano
William P. Fox
Steven B. Horton

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Mathematical Modeling

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Frank R. Giordano, William P. Fox and Steven B. Horton

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Contents

2 The Modeling Process, Proportionality, and Geometric Similarity 58

3 Model Fitting 105

4 Experimental Modeling 137

7 Optimization of Discrete Models 242

8 Modeling Using Graph Theory 298

Additional Contents on Website

1 UMAP Modules (COMAP)

The Undergraduate Applications in Mathematics modules (UMAPs) are developed and produced by the Consortium for Mathematics and Its Applications, Inc. (800-772-6627, **www.comap.com**). UMAPs are particularly suited to supplement the modeling course we propose. The following UMAPs are referenced as projects, further reading, or sources for additional problems and are provided on the CD for easy access.

UMAP 60–62 *The Distribution of Resources*
UMAP 67 *Modeling the Nervous System*
UMAP 69 *The Digestive Process of Sheep*
UMAP 70 *Selection in Genetics*
UMAP 73 *Epidemics*
UMAP 74 *Tracer Methods in Permeability*
UMAP 75 *Feldman's Model*
UMAP 208 *General Equilibrium: I*
UMAP 211 *The Human Cough*
UMAP 232 *Kinetics of Single-Reactant Reactions*
UMAP 234 *Radioactive Chains: Parents and Daughters*
UMAP 269 *Monte Carlo: The Use of Random Digits*
UMAP 270 *Lagrange Multipliers: Applications to Economics*
UMAPs 292–293 *Listening to the Earth: Controlled Source Seismology*
UMAP 294 *Price Discrimination and Consumer Surplus*
UMAP 303 *The Diffusion of Innovation in Family Planning*
UMAP 304 *Growth of Partisan Support I*
UMAP 305 *Growth of Partisan Support II*
UMAP 308 *The Richardson Arms Race Model*
UMAP 311 *The Geometry of the Arms Race*
UMAP 321 *Curve Fitting via the Criterion of Least Squares*
UMAP 322 *Difference Equations with Applications*
UMAP 327 *Adjusted Rates: The Direct Rate*
UMAP 331 *Ascent-Descent*
UMAP 332 *The Budgetary Process I*
UMAP 333 *The Budgetary Process II*
UMAP 340 *The Poisson Random Process*
UMAP 341 *Five Applications of Max-Min Theory from Calculus*
UMAP 376 *Differentiation, Curve Sketching, and Cost Functions*
UMAP 453 *Linear Programming in Two Dimensions: I*
UMAP 454 *Linear Programming in Two Dimensions: II*
UMAP 468 *Calculus of Variations with Applications in Mechanics*
UMAP 506 *The Relationship Between Directional Heading of an Automobile and Steering Wheel Deflection*
UMAP 517 *Lagrange Multipliers and the Design of Multistage Rockets*
UMAP 518 *Oligopolistic Competition*
UMAP 520 *Random Walks: An Introduction to Stochastic Processes*
UMAP 522 *Unconstrained Optimization*
UMAP 526 *Dimensional Analysis*
UMAP 539 *I Will If You Will… Individual Threshold and Group Behavior*

UMAP 551 *The Pace of Life: An Introduction to Empirical Model Fitting*
UMAP 564 *Keeping Dimensions Straight*
UMAP 590 *Random Numbers*
UMAP 610 *Whales and Krill: A Mathematical Model*
UMAP 628 *Competitive Hunter Models*
UMAP 675 *The Lotka-Volterra Predator-Prey Model*
UMAP 684 *Linear Programming via Elementary Matrices*
UMAP 709 *A Blood Cell Population Model, Dynamical Diseases, and Chaos*
UMAP 737 *Geometric Programming*
UMAP 738 *The Hardy-Weinberg Equilibrium*

2 Past Modeling Contest Problems

Past contest problems are excellent sources for modeling projects or sources to design a problem. On the CD we provide links to electronic copies of all contest problems:

Mathematical Contest in Modeling (MCM): *1985–2012*
Interdisciplinary Contest in Modeling (ICM): *1997–2012*
High School Contest in Modeling (HiMCM): *1998–2012*

3 Interdisciplinary Lively Applications Projects (ILAPs)

Interdisciplinary Lively Applications Projects (ILAPs) are developed and produced by the Consortium for Mathematics and Its Applications, Inc., COMAP (800-772-6627, **www.comap.com**). ILAPs are codesigned with a partner discipline to provide in-depth model development and analysis from both a mathematical perspective and that of the partner discipline. We find the following ILAPs to be particularly well suited for the course we propose:

- Car Financing
- Choloform Alert
- Drinking Water
- Electric Power
- Forest Fires
- Game Theory
- Getting the Salt Out
- Health Care
- Health Insurance Premiums
- Hopping Hoop
- Bridge Analysis
- Lagniappe Fund
- Lake Pollution
- Launch the Shuttle
- Pollution Police
- Ramps and Freeways
- Red & Blue CDs
- Drug Poisoning
- Shuttle
- Stocking a Fish Pond
- Survival of Early Americans
- Traffic Lights
- Travel Forecasting
- Tuition Prepayment
- Vehicle Emissions
- Water Purification

4 Technology and Software

Mathematical modeling often requires technology in order to use the techniques discussed in the text, the modules, and ILAPs. We provide extensive examples of technology using spreadsheets (Excel), computer algebra systems (Maple®, Mathematica®, and Matlab®), and the graphing calculator (TI). Application areas include:

- Difference Equations
- Model Fitting
- Empirical Model Construction
- Divided Difference Tables
- Cubic Splines
- Monte Carlo Simulation Models
- Discrete Probabilistic Models
- Reliability Models
- Linear Programming
- Golden Section Search
- Euler's Method for Ordinary Differential Equations
- Euler's Method for Systems of Ordinary Differential Equations
- Nonlinear Optimization

5 Technology Labs

Examples and exercises designed for student use in a laboratory environment are included, addressing the following topics:

- Difference Equations
- Proportionality
- Model Fitting
- Empirical Model Construction
- Monte Carlo Simulation
- Linear Programming
- Discrete Optimization Search Methods
- Ordinary Differential Equations
- Systems of Ordinary Differential Equations
- Continuous Optimization Search Methods

Preface

To facilitate an early initiation of the modeling experience, the first edition of this text was designed to be taught concurrently or immediately after an introductory business or engineering calculus course. In the second edition, we added chapters treating discrete dynamical systems, linear programming and numerical search methods, and an introduction to probabilistic modeling. Additionally, we expanded our introduction of simulation. In the third edition we included solution methods for some simple dynamical systems to reveal their long-term behavior. We also added basic numerical solution methods to the chapters covering modeling with differential equations. In the fourth edition, we added a new chapter to address modeling using graph theory, an area of burgeoning interest for modeling contemporary scenarios. Our chapter introduces graph theory from a modeling perspective to encourage students to pursue the subject in greater detail. We also added two new sections to the chapter on modeling with a differential equation: discussions of separation of variables and linear equations. Many of our readers had expressed a desire that analytic solutions to first-order differential equations be included as part of their modeling course. In the fifth edition, we have added two new chapters, Chapter 9, Modeling with Decision Theory and Chapter 10, Game Theory. Decision theory, also called decision analysis, is a collection of mathematical models to assist people in choosing among alternative courses of action in complex situations involving chance and risk. Game Theory then expands decision theory to include decisions where the payoff for the decision maker depends upon one or more additional decision makers. We present both total and partial conflict games.

The text is organized into two parts: Part One, Discrete Modeling (Chapters 1–10 and Chapter 14), and Part Two, Continuous Modeling (Chapters 11–13 and Chapter 15). This organizational structure allows for teaching an entire modeling course that is based on Part One and does not require the calculus. Part Two then addresses continuous models including models requiring optimization and models using differential equations that can be presented concurrently with freshman calculus. The text gives students an opportunity to cover all phases of the mathematical modeling process. The CD-ROM accompanying the text contains software, additional modeling scenarios and projects, and a link to past problems from the Mathematical Contest in Modeling. We thank Sol Garfunkel and the COMAP staff for their support of modeling activities that we refer to under Resource Materials below.

Goals and Orientation

The course continues to be a bridge between the study of mathematics and the applications of mathematics to various fields. The course affords the student an early opportunity to see how the pieces of an applied problem fit together. The student investigates meaningful and practical problems chosen from common experiences encompassing many academic disciplines, including the mathematical sciences, operations research, engineering, and the management and life sciences.

This text provides an introduction to the entire modeling process. Students will have opportunities to practice the following facets of modeling and enhance their problem-solving capabilities:

1. *Creative and Empirical Model Construction:* Given a real-world scenario, the student learns to identify a problem, make assumptions and collect data, propose a model, test the assumptions, refine the model as necessary, fit the model to data if appropriate, and analyze the underlying mathematical structure of the model to appraise the sensitivity of the conclusions when the assumptions are not precisely met.
2. *Model Analysis:* Given a model, the student learns to work backward to uncover the implicit underlying assumptions, assess critically how well those assumptions fit the scenario at hand, and estimate the sensitivity of the conclusions when the assumptions are not precisely met.
3. *Model Research:* The student investigates a specific area to gain a deeper understanding of some behavior and learns to use what has already been created or discovered.

Student Background and Course Content

Because our desire is to initiate the modeling experience as early as possible in the student's program, the only prerequisite for Chapters 11, 12 and 13 is a basic understanding of single-variable differential and integral calculus. Although some unfamiliar mathematical ideas are taught as part of the modeling process, the emphasis is on using mathematics that the students already know after completing high school. This is especially true in Part One. The modeling course will then motivate students to study the more advanced courses such as linear algebra, differential equations, optimization and linear programming, numerical analysis, probability, and statistics. The power and utility of these subjects are intimated throughout the text.

Further, the scenarios and problems in the text are not designed for the application of a particular mathematical technique. Instead, they demand thoughtful ingenuity in using fundamental concepts to find reasonable solutions to ''open-ended'' problems. Certain mathematical techniques (such as Monte Carlo simulation, curve fitting, and dimensional analysis) are presented because often they are not formally covered at the undergraduate level. Instructors should find great flexibility in adapting the text to meet the particular needs of students through the problem assignments and student projects. We have used this material to teach courses to both undergraduate and graduate students—and even as a basis for faculty seminars.

Organization of the Text

The organization of the text is best understood with the aid of Figure 1. The first ten chapters (and Chapter 14) constitute Part One and, require only precalculus mathematics as a

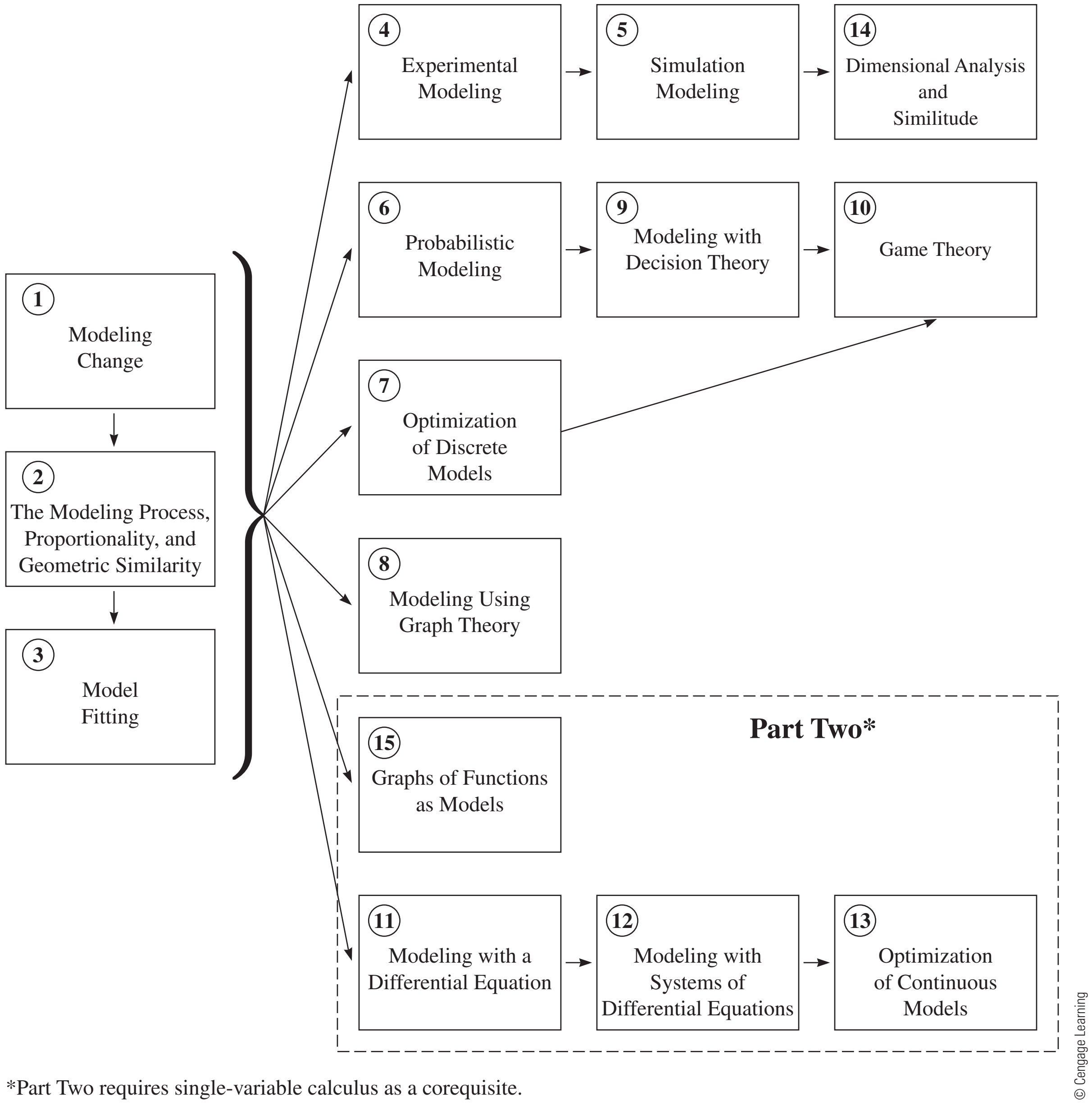


Figure 1

Chapter organization and progression

prerequisite. We begin with the idea of modeling *change* using simple finite difference equations. This approach is quite intuitive to the student and provides us with several concrete models to support our discussion of the modeling process in Chapter 2. There we classify models, analyze the modeling process, and construct several proportionality models or submodels that are then revisited in the next two chapters. In Chapter 3 the student is presented with three criteria for fitting a specific type of curve to a collected data set, with emphasis on the least-squares criterion. Chapter 4 addresses the problem of capturing the trend of a collected set of data. In this empirical construction process, we begin with fitting simple one-term models approximating collected data sets and then progress to more sophisticated interpolating models, including polynomial smoothing models and cubic splines. Simulation models are discussed in Chapter 5. An empirical model is fit to some collected data, and then Monte Carlo simulation is used to duplicate the behavior being investigated. The presentation motivates the eventual study of probability and statistics.

Chapter 6 provides an introduction to probabilistic modeling. The topics of Markov processes, reliability, and linear regression are introduced, building on scenarios and analysis presented previously. Chapter 7 addresses the issue of finding the best-fitting model using the other two criteria presented in Chapter 3. Linear programming is the method used for finding the best model for one of the criteria, and numerical search techniques can be used for the other. The chapter concludes with an introduction to numerical search methods, including the Dichotomous and Golden Section methods. Chapters 9 and 10 treat decision making under risk and uncertainty, with either one decision maker (Chapter 9) or two or more decision makers (Chapter 10). Part One then skips to Chapter 14, which is devoted to dimensional analysis, a topic of great importance in the physical sciences and engineering.

Part Two is dedicated to the study of continuous models. In Chapters 11 and 12 we model dynamic (time varying) scenarios. These chapters build on the discrete analysis presented in Chapter 1 by now considering situations where time is varying continuously. Chapter 13 is devoted to the study of continuous optimization. Chapter 15 treats the construction of continuous graphical models and explores the sensitivity of the models constructed to the assumptions underlying them. Students get the opportunity to solve continuous optimization problems requiring only the application of elementary calculus and are introduced to constrained optimization problems as well.

Student Projects

Student projects are an essential part of any modeling course. This text includes projects in creative and empirical model construction, model analysis, and model research. Thus we recommend a course consisting of a mixture of projects in all three facets of modeling. These projects are most instructive if they address scenarios that have no unique solution. Some projects should include *real* data that students are either given or can *readily* collect. A combination of individual and group projects can also be valuable. Individual projects are appropriate in those parts of the course in which the instructor wishes to emphasize the development of individual modeling skills. However, the inclusion of a group project early in the course gives students the exhilaration of a ‘‘brainstorming’’ session. A variety of projects is suggested in the text, such as constructing models for various scenarios, completing UMAP[1] modules, or researching a model presented as an example in the text or class. It is valuable for each student to receive a mixture of projects requiring either model construction, model analysis, or model research for variety and confidence building throughout the course. Students might also choose to develop a model in a scenario of particular interest, or analyze a model presented in another course. We recommend five to eight short projects in a typical modeling course. Detailed suggestions on how student projects can be assigned and used are included in the Instructor's Manual that accompany this text.

In terms of the number of scenarios covered throughout the course, as well as the number of homework problems and projects assigned, we have found it better to pursue a few that are developed carefully and completely. We have provided many more problems and projects than can reasonably be assigned to allow for a wide selection covering many different application areas.

Resource Materials

We have found material provided by the Consortium for Mathematics and Its Application (COMAP) to be outstanding and particularly well suited to the course we propose. Individual modules for the undergraduate classroom, UMAP Modules, may be used in a variety of ways. First, they may be used as instructional material to support several lessons. In this mode a student completes the self-study module by working through its exercises (the detailed solutions provided with the module can be conveniently removed before it is issued). Another option is to put together a block of instruction using one or more UMAP modules suggested in the projects sections of the text. The modules also provide excellent sources for ''model research,'' because they cover a wide variety of applications of mathematics in many fields. In this mode, a student is given an appropriate module to research and is asked to complete and report on the module. Finally, the modules are excellent resources for scenarios for which students can practice model construction. In this mode the instructor writes a scenario for a student project based on an application addressed in a particular module and uses the module as background material, perhaps having the student complete the module at a later date. The CD accompanying the text contains most of the UMAPs referenced throughout. Information on the availability of newly developed interdisciplinary projects can be obtained by writing COMAP at the address given previously, calling COMAP at 1-800-772-6627, or electronically: order@comap.com.

A great source of student-group projects are the Mathematical Contest in Modeling (MCM) and the Interdisciplinary Contest in Modeling (ICM). These projects can be taken from the link provided on the CD and tailored by the instructor to meet specific goals for their class. These are also good resources to prepare teams to compete in the MCM and ICM contests. The contest is sponsored by COMAP with funding support from the National Security Agency, the Society of Industrial and Applied Mathematics, the Institute

[1] UMAP modules are developed and distributed through COMAP, Inc., 57 Bedford Street, Suite 210, Lexington, MA 02173.

for Operations Research and the Management Sciences, and the Mathematical Association of America. Additional information concerning the contest can be obtained by contacting COMAP, or visiting their website at **www.comap.com**.

The Role of Technology

Technology is an integral part of doing mathematical modeling with this textbook. Technology can be used to support the modeling of solutions in all of the chapters. Rather than incorporating lots of varied technologies into the explanations of the models directly in the text, we decided to include the use of various technology on the enclosed CD. There the student will find templates in Microsoft® Excel®, Maple®, Mathematica®, and Texas Instruments graphing calculators, including the TI-83 and 84 series.

We have chosen to illustrate the use of *Maple* in our discussion of the following topics that are well supported by Maple commands and programming procedures: difference equations, proportionality, model fitting (least squares), empirical models, simulation, linear programming, dimensional analysis, modeling with differential equations, modeling with systems of differential equations, and optimization of continuous models. Maple worksheets for the illustrative examples appearing in the referenced chapters are provided on the CD.

Mathematica was chosen to illustrate difference equations, proportionality, model fitting (least squares), empirical models, simulation, linear programming, graph theory, dimensional analysis, modeling with differential equations, modeling with systems of differential equations, and optimization of continuous models. Mathematica worksheets for illustrative examples in the referenced chapters are provided on the CD.

Excel is a spreadsheet that can be used to obtain numerical solutions and conveniently obtain graphs. Consequently, Excel was chosen to illustrate the iteration process and graphical solutions to difference equations. It was also selected to calculate and graph functions in proportionality, model fitting, empirical modeling (additionally used for divided difference tables and the construction and graphing of cubic splines), Monte Carlo simulation, linear programming (Excel's Solver is illustrated), modeling with differential equations (numerical approximations with both the Euler and the Runge-Kutta methods), modeling with systems of differential equations (numerical solutions), and Optimization of Discrete and Continuous Models (search techniques in single-variable optimization such as the dichotomous and Golden Section searches). Excel worksheets can be found on the website.

The *TI calculator* is a powerful tool for technology as well. Much of this textbook can be covered using the TI calculator. We illustrate the use of TI calculators with difference equations, proportionality, modeling fitting, empirical models (Ladder of Powers and other transformations), simulation, and differential equations (Euler's method to construct numerical solutions).

Acknowledgments

It is always a pleasure to acknowledge individuals who have played a role in the development of a book. We are particularly grateful to Brigadier General (retired) Jack M. Pollin and Dr. Carroll Wilde for stimulating our interest in teaching modeling and for support and guidance in our careers. We acknowledge and are especially grateful for the support and encouragement of Sol Garfunkel and the entire COMAP staff who have been especially helpful in all five editions: they are true pioneers and champions of mathematical modeling at all levels. We're indebted to many colleagues for reading the first edition manuscript and suggesting modifications and problems: Rickey Kolb, John Kenelly, Robert Schmidt, Stan Leja, Bard Mansager, and especially Steve Maddox and Jim McNulty. We especially thank Maurice D. Weir for his contribution as an author to previous editions. We also especially thank Richard West for his role as a reviewer for the fifth edition.

We are indebted to a number of individuals who authored or co-authored UMAP materials that support the text: David Cameron, Brindell Horelick, Michael Jaye, Sinan Koont, Stan Leja, Michael Wells, and Carroll Wilde. In addition, we thank Solomon Garfunkel and the entire COMAP staff for their cooperation on all five editions of this text. We also thank Tom O'Neil and his students for their contributions to the CD and for helpful suggestions in support of modeling activities. We would like to thank Dr. Amy H. Erickson for her many contributions to the CD and website.

Thanks to the following reviewers of the fifth edition: John Dossey, Robert Burks, and Richard West.

The production of any mathematics text is a complex process, and we have been especially fortunate in having a superb and creative production staff at Brooks/Cole and Cengage for the production of each edition of the text. We would like to thank everyone from Cengage who worked with us on this edition, especially Molly Taylor, our Acquisitions Editor, and Shaylin Walsh-Hogan, our Assistant Editor. Thanks also to Prashanth Kamavarapu and PreMediaGlobal for production service.

Frank R. Giordano
William P. Fox
Steven B. Horton

Dedicated to our mentors, colleagues, and students, who continue to motivate mathematical modeling in our lives, our teaching, and our research,

Especially to

Jack M. Pollin

Leader, Teacher, Scholar, Solider, Mentor, and Friend

Painting by Jack M. Pollin, BG(Ret), US Army

Founding Father of Mathematical Modeling at West Point

1 Modeling Change

Introduction

To help us better understand our world, we often describe a particular phenomenon mathematically (by means of a function or an equation, for instance). Such a **mathematical model** is an idealization of the real-world phenomenon and never a completely accurate representation. Although any model has its limitations, a good one can provide valuable results and conclusions. In this chapter we direct our attention to modeling change.

Mathematical Models

In modeling our world, we are often interested in predicting the value of a variable at some time in the future. Perhaps it is a population, a real estate value, or the number of people with a communicative disease. Often a mathematical model can help us understand a behavior better or aid us in planning for the future. Let's think of a mathematical model as a mathematical construct designed to study a particular real-world system or behavior of interest. The model allows us to reach mathematical conclusions about the behavior, as illustrated in Figure 1.1. These conclusions can be interpreted to help a decision maker plan for the future.

Simplification

Most models simplify reality. Generally, models can only approximate real-world behavior. One very powerful simplifying relationship is **proportionality**.

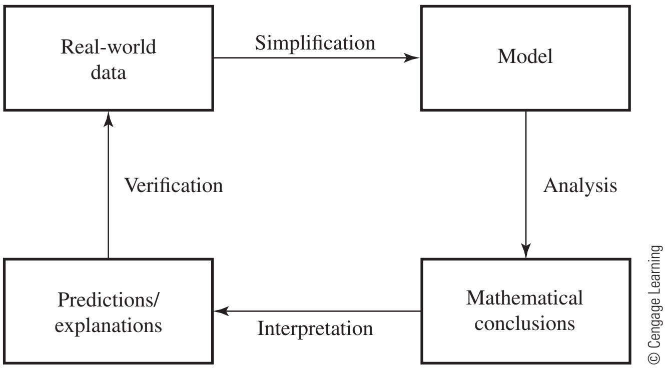


Figure 1.1
A flow of the modeling process beginning with an examination of real-world data

Definition

Two variables y and x are **proportional** (to each other) if one is always a constant multiple of the other—that is, if

$$y = kx$$

for some nonzero constant k. We write $y \propto x$.

The definition means that the graph of y versus x lies along a straight line through the origin. This graphical observation is useful in testing whether a given data collection reasonably assumes a proportionality relationship. If a proportionality is reasonable, a plot of one variable against the other should approximate a straight line through the origin. Here is an example.

EXAMPLE 1 *Testing for Proportionality*

Table 1.1
Spring–mass system

Mass	Elong.
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
300	4.875
350	5.675
400	6.500
450	7.250
500	8.000
550	8.750

Consider a spring–mass system, such as the one shown in Figure 1.2. We conduct an experiment to measure the stretch of the spring as a function of the mass (measured as weight) placed on the spring. Consider the data collected for this experiment, displayed in Table 1.1. A scatterplot graph of the stretch or elongation of the spring versus the mass or weight placed on it reveals a straight line passing approximately through the origin (Figure 1.3).

The data appear to follow the proportionality rule that elongation e is proportional to the mass m, or, symbolically, $e \propto m$. The straight line appears to pass through the origin. This geometric understanding allows us to look at the data to determine whether proportionality is a reasonable simplifying assumption and, if so, to estimate the slope k. In this case, the assumption appears valid, so we estimate the constant of proportionality by picking the two points (200, 3.25) and (300, 4.875) as lying along the straight line. We calculate the slope of the line joining these points as

$$slope = \frac{4.875 - 3.25}{300 - 200} = 0.01625$$

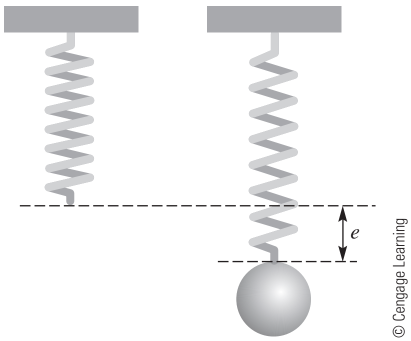
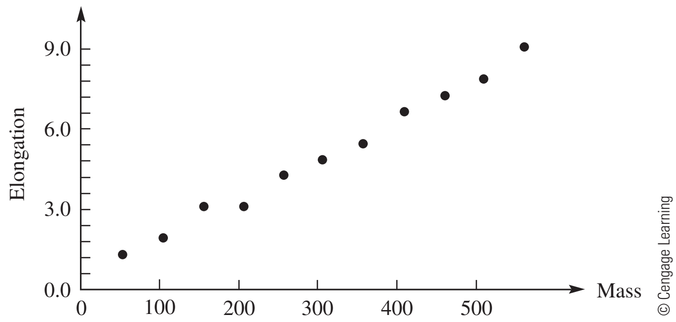


Figure 1.2
Spring–mass system

Figure 1.3
Data from spring–mass system

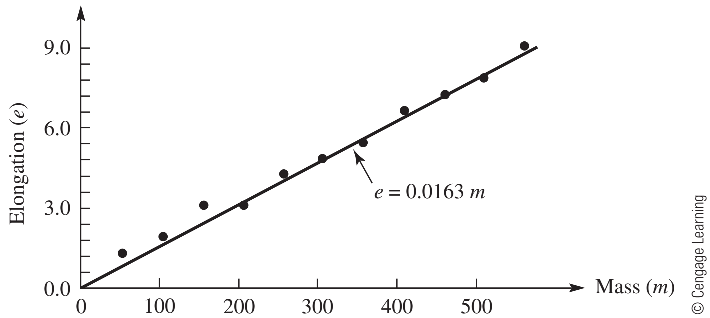


Thus the constant of proportionality is approximately 0.0163, the slope of the line through the origin. We estimate our model as

$$e = 0.0163m$$

We then examine how close our model fits the data by plotting the line it represents superimposed on the scatterplot (Figure 1.4). The graph reveals that the simplifying proportionality model is reasonable, because most of the points fall on or very near the line $e = 0.0163m$.

Figure 1.4
Data from spring–mass system with proportionality line



Modeling Change

A powerful paradigm to use in modeling change is

$$\textit{future value} = \textit{present value} + \textit{change}$$

Often, we wish to predict the future on the basis of what we know now, in the present, and add the change that has been carefully observed. In such cases, we begin by studying the change itself according to the formula

$$\textit{change} = \textit{future value} - \textit{present value}$$

By collecting data over a period of time and plotting those data, we often can discern patterns to model that capture the trend of the change. If the behavior is taking place over *discrete time periods*, the preceding construct leads to a **difference equation**, which we study in this chapter. If the behavior is taking place *continuously* with respect to time, then the construct leads to a **differential equation** studied in Chapter 11. Both are powerful methodologies for studying change to explain and predict behavior.

1.1 Modeling Change with Difference Equations

In this section we build mathematical models to describe change in an observed behavior. When we observe change, we are often interested in understanding why the change occurs in the way it does, perhaps to analyze the effects of different conditions on the behavior or to predict what will happen in the future. A mathematical model helps us better understand a behavior, while allowing us to experiment mathematically with different conditions affecting it.

Definition

For a sequence of numbers $A = \{a_0, a_1, a_2, a_3, \ldots\}$ the first differences are

$$\begin{aligned}\Delta a_0 &= a_1 - a_0\\ \Delta a_1 &= a_2 - a_1\\ \Delta a_2 &= a_3 - a_2\\ \Delta a_3 &= a_4 - a_3\end{aligned}$$

For each positive integer n, the **nth first difference** is

$$\Delta a_n = a_{n+1} - a_n$$

Note from Figure 1.5 that the first difference represents the rise or fall between consecutive values of the sequence—that is, the vertical *change* in the graph of the sequence during one time period.

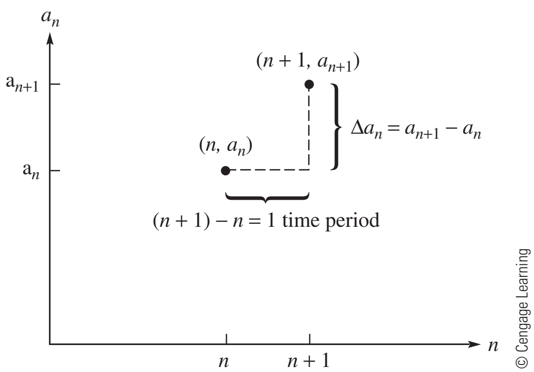


Figure 1.5
The first difference of a sequence is the rise in the graph during one time period.

EXAMPLE 1 *A Savings Certificate*

Consider the value of a savings certificate initially worth \$1000 that accumulates interest paid each month at 1% per month. The following sequence of numbers represents the value of the certificate month by month.

$$A = (1000, 1010, 1020.10, 1030.30, \ldots)$$

The first differences are as follows:

$$\begin{aligned} \Delta a_0 &= a_1 - a_0 = 1010 - 1000 = 10 \\ \Delta a_1 &= a_2 - a_1 = 1020.10 - 1010 = 10.10 \\ \Delta a_2 &= a_3 - a_2 = 1030.30 - 1020.10 = 10.20 \end{aligned}$$

Note that the first differences represent the *change in the sequence* during one time period, or the *interest earned* in the case of the savings certificate example.

The first difference is useful for modeling change taking place in discrete intervals. In this example, the change in the value of the certificate from one month to the next is merely the interest paid during that month. If n is the number of months and a_n the value of the certificate after n months, then the change or interest growth in each month is represented by the nth difference

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n$$

This expression can be rewritten as the difference equation

$$a_{n+1} = a_n + 0.01a_n$$

We also know the initial deposit of \$1000 (initial value) that then gives the **dynamical system model**

$$\begin{aligned} a_{n+1} &= 1.01a_n, \quad n = 0, 1, 2, 3, \ldots \\ a_0 &= 1000 \end{aligned} \tag{1.1}$$

where a_n represents the amount accrued after n months. Because n represents the nonnegative integers $\{0, 1, 2, 3, \ldots\}$, Equation (1.1) represents an *infinite set* of algebraic equations, called a **dynamical system**. Dynamical systems allow us to describe the *change* from one period to the next. The difference equation formula computes the next term, given the immediately previous term in the sequence, but it does not compute the value of a specific term directly (e.g., the savings after 100 periods). We would iterate the sequence to a_{100} to obtain that value.

Because it is change we often observe, we can construct a difference equation by representing or approximating the change from one period to the next. To modify our example, if we were to withdraw \$50 from the account each month, the change during a period would be the interest earned during that period minus the monthly withdrawal, or

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n - 50$$

In most examples, mathematically describing the change is not going to be as precise a procedure as illustrated here. Often it is necessary to *plot the change*, *observe a pattern*, and then *describe the change* in mathematical terms. That is, we will be trying to find

$$\textit{change} = \Delta a_n = \textit{some function } f$$

The change may be a function of previous terms in the sequence (as was the case with no monthly withdrawals), or it may also involve some external terms (such as the amount of money withdrawn in the current example or an expression involving the period n). Thus, in constructing models representing change in this chapter, we will be modeling change in discrete intervals, where

$$\textit{change} = \Delta a_n = a_{n+1} - a_n = f(\textit{terms in the sequence, external terms})$$

Modeling change in this way becomes the art of determining or approximating a function f that represents the change.

Consider a second example in which a difference equation exactly models a behavior in the real world.

EXAMPLE 2 *Mortgaging a Home*

Six years ago your parents purchased a home by financing \$80,000 for 20 years, paying monthly payments of \$880.87 with a monthly interest of 1%. They have made 72 payments and wish to know how much they owe on the mortgage, which they are considering paying off with an inheritance they received. Or they could be considering refinancing the mortgage with several interest rate options, depending on the length of the payback period. The change in the amount owed each period increases by the amount of interest and decreases by the amount of the payment:

$$\Delta b_n = b_{n+1} - b_n = 0.01 b_n - 880.87$$

Solving for b_{n+1} and incorporating the initial condition gives the dynamical system model

$$\begin{aligned} b_{n+1} &= b_n + 0.01 b_n - 880.87 \\ b_0 &= 80000 \end{aligned}$$

where b_n represents the amount owed after n months. Thus,

$$\begin{aligned} b_1 &= 80000 + 0.01(80000) - 880.87 = 79919.13 \\ b_2 &= 79919.13 + 0.01(79919.13) - 880.87 = 79837.45 \end{aligned}$$

yielding the sequence

$$B = (80000, 79919.13, 79837.45, \ldots)$$

Calculating b_3 from b_2, b_4 from b_3, and so forth in turn, we obtain $b_{72} = \$71{,}523.11$. The sequence is graphed in Figure 1.6.

Months n	Amount owed b_n
0	80000.00
1	79919.13
2	79837.45
3	79754.96
4	79671.64
5	79587.48
6	79502.49
7	79416.64
8	79329.94
9	79242.37
10	79153.92
11	79064.59
12	78974.37

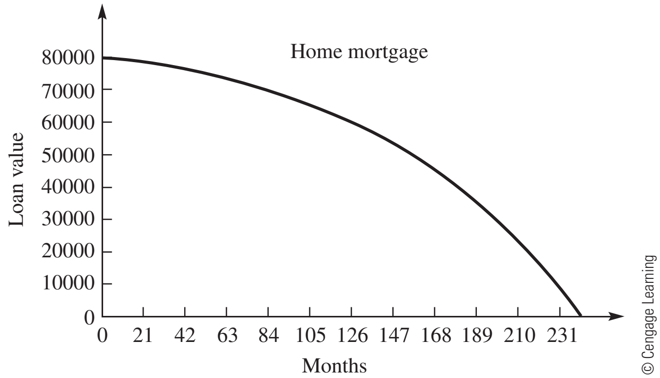


Figure 1.6
The sequence and graph for Example 2

Let's summarize the important ideas introduced in Examples 1 and 2.

Definition

A **sequence** is a function whose domain is the set of all nonnegative integers and whose range is a subset of the real numbers. A **dynamical system** is a relationship among terms in a sequence. A **numerical solution** is a table of values satisfying the dynamical system.

In the problems for this section we discuss other behaviors in the world that can be modeled exactly by difference equations. In the next section, we use difference equations to approximate observed change. After collecting data for the change and discerning patterns of the behavior, we will use the concept of proportionality to test and fit models that we propose.

1.1 PROBLEMS

SEQUENCES

1. Write out the first five terms a_0–a_4 of the following sequences:

a. $a_{n+1} = 3a_n, \quad a_0 = 1$

b. $a_{n+1} = 2a_n + 6, \quad a_0 = 0$

c. $a_{n+1} = 2a_n(a_n + 3), \quad a_0 = 4$

d. $a_{n+1} = a_n^2, \quad a_0 = 1$

2. Find a formula for the nth term of the sequence.

a. $\{3, 3, 3, 3, 3, \ldots\}$

b. $\{1, 4, 16, 64, 256, \ldots\}$

c. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \ldots\}$

d. $\{1, 3, 7, 15, 31, \ldots\}$

DIFFERENCE EQUATIONS

3. By examining the following sequences, write a difference equation to represent the change during the nth interval as a function of the previous term in the sequence.

a. $\{2, 4, 6, 8, 10, \ldots\}$

b. $\{2, 4, 16, 256, \ldots\}$

c. $\{1, 2, 5, 11, 23, \ldots\}$

d. $\{1, 8, 29, 92, \ldots\}$

4. Write out the first five terms of the sequence satisfying the following difference equations:

a. $\Delta a_n = \frac{1}{2}a_n, \quad a_0 = 1$

b. $\Delta b_n = 0.015b_n, \quad b_0 = 1000$

c. $\Delta p_n = 0.001(500 - p_n), \quad p_0 = 10$

d. $\Delta t_n = 1.5(100 - t_n), \quad t_0 = 200$

DYNAMICAL SYSTEMS

5. By substituting $n = 0, 1, 2, 3$, write out the first four algebraic equations represented by the following dynamical systems:

a. $a_{n+1} = 3a_n, \quad a_0 = 1$

b. $a_{n+1} = 2a_n + 6, \quad a_0 = 0$

c. $a_{n+1} = 2a_n(a_n + 3), \quad a_0 = 4$

d. $a_{n+1} = a_n^2, \quad a_0 = 1$

6. Name several behaviors you think can be modeled by dynamical systems.

MODELING CHANGE EXACTLY

For Problems 7–10, formulate a dynamical system that models change exactly for the described situation.

7. You currently have \$5000 in a savings account that pays 0.5% interest each month. You add another \$200 each month.

8. You owe \$500 on a credit card that charges 1.5% interest each month. You pay \$50 each month and you make no new charges.

9. Your parents are considering a 30-year, \$200,000 mortgage that charges 0.5% interest each month. Formulate a model in terms of a monthly payment p that allows the mortgage (loan) to be paid off after 360 payments. *Hint*: If a_n represents the amount owed after n months, what are a_0 and a_{360}?

10. Your grandparents have an annuity. The value of the annuity increases each month by an automatic deposit of 1% interest on the previous month's balance. Your grandparents withdraw \$1000 at the beginning of each month for living expenses. Currently, they have \$50,000 in the annuity. Model the annuity with a dynamical system. Will the annuity run out of money? When? *Hint*: What value will a_n have when the annuity is depleted?

11. Repeat Problem 10 with an interest rate of 0.5%.

12. Your current credit card balance is $12,000 with a current rate of 19.9% per year. Interest is charged monthly. Determine what monthly payment p will pay off the card in

a. Two years, assuming no new charges

b. Four years, assuming no new charges

13. Again consider Problem 12 above. Now assume that each month you charge $105. Determine what monthly payment p will pay off the card in

a. Two years

b. Four years

1.1 PROJECTS

1. With the price of gas continuing to rise, you wish to look at cars that get better gas mileage. You narrow down your choices to the following 2012 models: Ford Fiesta, Ford Focus, Chevy Volt, Chevy Cruz, Toyota Camry, Toyota Camry Hybrid, Toyota Prius and Toyota Corolla. Each company has offered you their "best deal" as listed in the following table. You are able to allocate approximately $500 for a car payment each month up to 60 months, although less time would be preferable. Use dynamical systems to determine which new car you can afford.

2012 Model	Best Deal Price	Cash Down	Interest and Duration
Ford Fiesta	$14,200	$500	4.5% APR for 60 months
Ford Focus	$20,705	$750	4.38% APR for 60 months
Chevy Volt	$39,312	$1,000	3.28% APR for 48 months
Chevy Cruz	$16,800	$500	4.4% APR for 60 months
Toyota Camry	$22,955	0	4.8% APR for 60 months
Toyota Camry Hybrid	$26,500	0	3% APR for 48 months
Toyota Corolla	$16,500	$900	4.25% for 60 months
Toyota Prius	$19,950	$1,000	4.3% for 60 months

2. You are considering a 30-year mortgage that charges 0.4% interest each month to pay off a $250,000 mortgage.

a. Determine the monthly payment p that allows the loan to be paid off at 360 months.

b. Now assume that you have been paying the mortgage for 8 years and now have an opportunity to refinance the loan. You have a choice between a 20-year loan at 4% per year with interest charged monthly and a 15-year loan at 3.8% per year with interest charged monthly. Each of the loans charges a closing cost of $2500. Determine the monthly payment p for both the 20-year loan and the 15-year loan. Do you think refinancing is the right thing to do? If so, do you prefer the 20-year or the 15-year option?